Technical Report No. 32-696

Dipolar Coordinates

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PASADENA, CALIFORNIA

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FOREWORD

This work was performed by Russell E. Carr as a part-time employee of the Jet Propulsion Laboratory while he was Assistant Director and Professor of Geophysics at the Geophysical Institute of the University of Alaska, College, Alaska. He is presently Head of the Department of Mathematics of the University of Alaska.

ABSTRACT

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Theoretical solutions to magnetohydrodynamics problems where a dipole magnetic field is assumed should provide more appropriate models for investigating the magnetosphere than solutions to problems where a uniform field has been used. One approach needing consideration is the use of dipolar coordinates.

A detailed derivation of dipolar coordinates is given, with a demonstration of how to express the commonly used vector formulas in dipolar coordinates. As a key part of this demonstration, it is shown how spherical coordinates can be expressed analytically in terms of dipolar coordinates.

It is concluded that, since the use of dipolar coordinates leads to a very unwieldy analytical description for the commonly used vector quantities, another approach might be more fruitful in attempting to solve magnetohydrodynamics problems in which a dipole field is assumed.

Author

I. INTRODUCTION

Dipolar coordinates were introduced by A. J. Dragt (Ref. 1) in a discussion of the effect of hydromagnetic waves on the lifetime of Van Allen radiation protons. Further elaboration on the use of dipolar coordinates was given by W. C. Hoffman (Ref. 2) in a discussion of the ionospheric wave equation.

Theoretical solutions to problems in magnetohydrodynamics have been limited largely to cases in which a uniform magnetic field has been assumed, but those concerned with the problems of the magnetosphere find it desirable to use a more appropriate model to represent the magnetic field. Since the Earth's magnetic field, apart from disturbances by the solar wind, is fairly well approximated by that of a dipole, it is natural to be concerned with the solution to problems where a dipole field is assumed. With such problems, one approach needing consideration is the use of dipolar coordinates.

It is the purpose of the present Report to give a detailed derivation of dipolar coordinates, to show how spherical coordinates can be expressed uniquely in terms of dipolar coordinates, and to present the basic vector formulas in the dipolar coordinate system.

II. THE COORDINATES

Dipolar coordinates are conveniently defined in terms of spherical coordinates; the latter are represented by r, ϕ, θ in the diagram in Fig. 1, where θ is the complement of the angle normally so labeled.

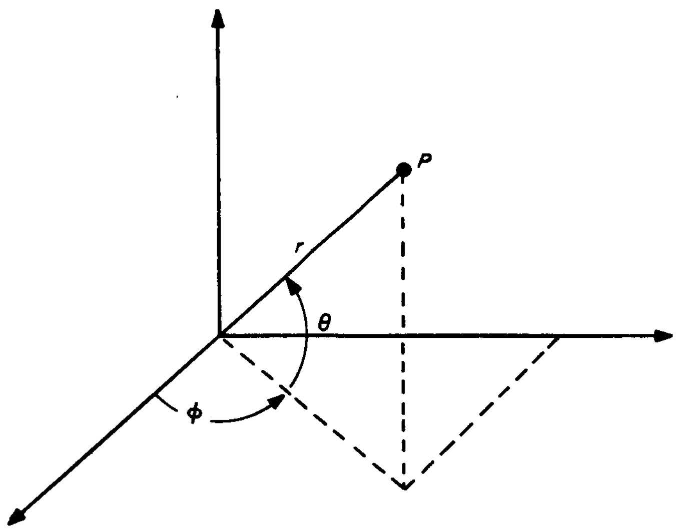

Fig. 1. The coordinates r, ϕ, θ

A dipole field of magnetic moment M may be thought of as the limiting case in which magnetic poles of strength m and $-m$, separated by a distance d, are brought toward one another with $M = md$ kept constant. A point P at a distance r from the midpoint is in a field with a potential (see Fig. 2):

$$V = \frac{m}{\left(r^2 + \frac{d^2}{4} - rd\sin\theta\right)^{1/2}} + \frac{-m}{\left(r^2 + \frac{d^2}{4} + rd\sin\theta\right)^{1/2}} \tag{1}$$

where θ is the complement of the angle subtended at the midpoint by P and the pole m. Replacing m by M/d equation (1) may be written as

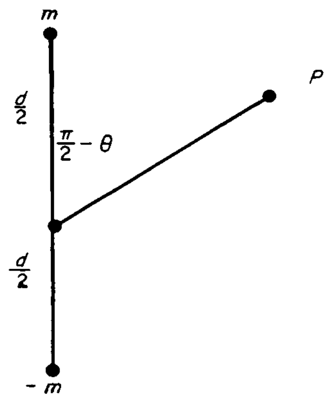

Fig. 2. A point in the field of two magnetic poles

which, when d/r is considered negligible, gives the potential for a dipole field

$$V = \frac{M\sin\theta}{r^2} \tag{2}$$

The magnetic force vector $\mathbf{H}$ at P is given by the negative of the gradient of V, so, with a dipole field,

$$\mathbf{H} = -\mathrm{grad}\left[\frac{M\sin\theta}{r^2}\right] \tag{3}$$

Since V and $\mathbf{H}$ are independent of ϕ, and since the gradient of V is normal to the surfaces $V =$ constant, the magnetic force $\mathbf{H}$ will be directed along lines of force which are the orthogonal trajectories to the equipotential curves

$$\begin{aligned} \phi &= \text{constant} \\ V &= \text{constant} \end{aligned} \tag{4}$$

From equation (2) it can be seen that the equipotential curves for a dipole field are given by

$$\begin{aligned} \phi &= \text{constant} \\ \frac{\sin\theta}{r^2} &= \text{constant} \end{aligned} \tag{5}$$

$$V = \frac{M}{d}\left\{\frac{\left(r + \frac{d}{2}\sin\theta\right)\left[1 + \frac{1}{2}\frac{\frac{d^2}{4}\cos^2\theta}{\left(r + \frac{d}{2}\sin\theta\right)^2} + \cdots\right] - \left(r - \frac{d}{2}\sin\theta\right)\left[1 - \frac{1}{2}\frac{\frac{d^2}{4}\cos^2\theta}{\left(r - \frac{d}{2}\sin\theta\right)^2} + \cdots\right]}{\left(r^2 + \frac{d^2}{4} - rd\sin\theta\right)^{1/2}\left(r^2 + \frac{d^2}{4} + rd\sin\theta\right)^{1/2}}\right\}$$

To find equations for the lines of force, substitute

$$\sin\theta = \frac{z}{(\rho^2 + z^2)^{1/2}}$$

$$r = (\rho^2 + z^2)^{1/2}$$

in

$$\frac{\sin\theta}{r^2} = \text{constant} \tag{6}$$

to obtain

$$(\rho^2 + z^2)^{3/2} = C_1 z$$

The differential equation for the family of curves (in the half-plane ϕ = constant) given by equation (6) is easily found to be

$$3\rho z\, d\rho - (\rho^2 - 2z^2)\, dz = 0 \tag{7}$$

The orthogonal trajectories are the family of curves with differential equation

$$(\rho^2 - 2z^2)\, d\rho + 3\rho z\, dz = 0 \tag{8}$$

Using the integrating factor $\rho^{-7/3}$, the solution to equation (8) is found to be

$$\rho^{-4/3}(\rho^2 + z^2) = C_2 \tag{9}$$

which transforms to

$$\frac{r}{\cos^2\theta} = \text{constant} \tag{10}$$

The orthogonal dipolar coordinates α, ϕ, β are then defined by

$$\alpha = \frac{r}{\cos^2\theta}$$
$$\phi = \phi \tag{11}$$
$$\beta = \frac{\sin\theta}{r^2}$$

The ranges on the three coordinates are

$$0 \leqq \alpha < \infty$$
$$0 \leqq \phi < 2\pi \tag{12}$$
$$-\infty < \beta < \infty$$

A network of curves, given (in the half-plane ϕ = constant) by α = constant, β = constant, is shown in Fig. 3. It is seen that, for fixed r, β increases as θ increases, and that $\beta = 0$ corresponds to $\theta = 0$. It is further seen that, for fixed θ, α increases as r increases, and that $\alpha = 0$ corresponds to the point $r = 0$. With ϕ = constant, lines of force are given by α = constant and equipotential lines are given by β = constant.

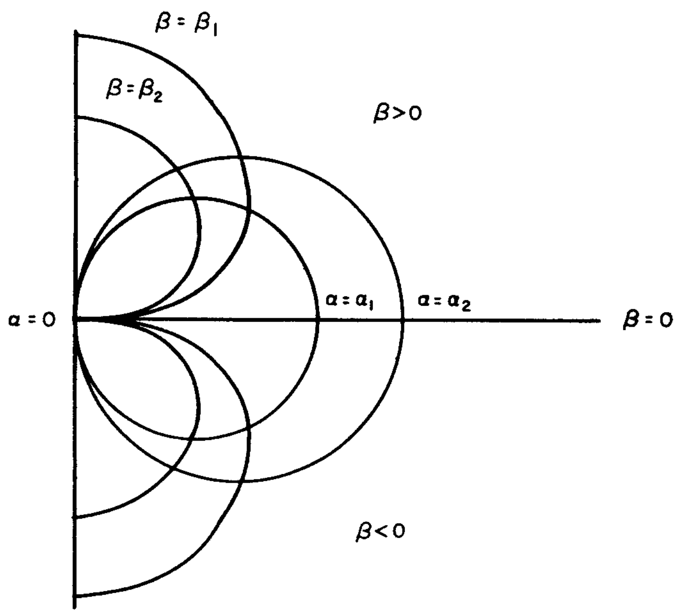

Fig. 3. The curves α = constant, β = constant in the half-plane, ϕ = constant

III. TRANSFORMATION TO SPHERICAL COORDINATES

The transformation from r, ϕ, and θ coordinates to α, ϕ, β coordinates is given by equations (11). The transformation from α, ϕ, β coordinates to r, ϕ, θ coordinates is more complicated, but yet uniquely defined for the ranges (12). Three cases must be treated:

Case 1: $\alpha = 0$

In this case, β is undefined; then $r = 0$ and θ is undefined.

Case 2: $\alpha > 0, \quad \beta = 0$

In this case, $\theta = 0$ and $r = \alpha$.

Case 3: $\alpha > 0, \quad \beta \neq 0$

Elimination of $\sin\theta$ from equations (11) gives the quartic equation in r,

$$r^4 + \frac{1}{\alpha\beta^2} r - \frac{1}{\beta^2} = 0 \tag{13}$$

while elimination of r gives the quartic equation in $\sin\theta$,

$$\sin^4\theta - 2\sin^2\theta - \frac{1}{\alpha^2\beta}\sin\theta + 1 = 0 \tag{14}$$

The solutions to equations (13) and (14) are special cases of the general solution to the quartic equation. Since the general quartic equation has four roots, it is necessary to find which root of each of the equations (13) and (14) gives the unique value for r or $\sin\theta$ corresponding to a particular choice of $\alpha > 0$, $\beta \neq 0$.

Writing equation (13) in the form

$$r^4 + \frac{1}{\alpha\beta^2} r = \frac{1}{\beta^2} \tag{13a}$$

it is clear that the left member increases from zero without limit as r increases from zero without limit, so there is one positive real root of equation (13).

Substituting x for $\sin\theta$, equation (14) may be written in the form

$$(x^2 - 1)^2 = \frac{1}{\alpha^2\beta} x \tag{14a}$$

Since, for real x, the left member is non-negative, x and β have the same sign. From graphical considerations, it can be seen that the straight line representing the right member of equation (14a) intersects the curve representing the left member in two different places. If $\beta > 0$, one value of x is between zero and one, the other greater than one. If $\beta < 0$, one value of x is between zero and minus one, the other less than minus one. Thus equation (14) gives one real value of θ, positive if $\beta > 0$ and negative if $\beta < 0$.

Uspensky (Ref. 3) treats the solution of the general quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0 \tag{15}$$

by reducing it to two quadratic equations

$$x^2 + \frac{a}{2}x + \frac{y}{2} = \pm(ex + f) \tag{16}$$

where

$$e = \left(\frac{a^2}{4} - b + y\right)^{1/2} \tag{17}$$

$$f = \pm\left(\frac{y^2}{4} - d\right)^{1/2} \tag{18}$$

the sign in equation (18) being chosen the same as that of

$$-c + \frac{ay}{2} \tag{19}$$

where y is any real root of the resolvent cubic

$$y^3 - by^2 + (ac - 4d)y + (4bd - a^2d - c^2) = 0 \tag{20}$$

Writing

$$A = -b, \qquad B = ac - 4d, \qquad C = 4bd - a^2d - c^2 \tag{21}$$

shows equation (20) to be a special case of the general cubic equation

$$y^3 + Ay^2 + By + C = 0 \tag{22}$$

The solution to equation (22) is given by

$$y = z - \frac{A}{3} \tag{23}$$

where

$$z^3 + pz + q = 0 \tag{24}$$

and

$$p = B - \frac{A^2}{3}, \qquad q = C - \frac{AB}{3} + \frac{2A^3}{27} \tag{25}$$

Then

$$z = u + v \tag{26}$$

where u is any cube root

$$u = \left[-\frac{q}{2} + \left(\frac{q^2}{4} + \frac{p^3}{27} \right)^{1/2} \right]^{1/3} \tag{27}$$

and v is the corresponding cube root

$$v = \left[-\frac{q}{2} - \left(\frac{q^2}{4} + \frac{p^3}{27} \right)^{1/2} \right]^{1/3} \tag{28}$$

such that

$$uv = -\frac{p}{3} \tag{29}$$

Applying Uspensky's solution (Ref. 3) to equation (13),

$$a = 0, \qquad b = 0, \qquad c = \frac{1}{\alpha\beta^2}, \qquad d = -\frac{1}{\beta^2} \tag{30}$$

so

$$A = 0, \qquad B = +\frac{4}{\beta^2}, \qquad C = -\frac{1}{\alpha^2\beta^4} \tag{31}$$

and

$$p = +\frac{4}{\beta^2}, \qquad q = -\frac{1}{\alpha^2\beta^4} \tag{32}$$

Then

$$u = \left[\frac{1}{2\alpha^2\beta^4} + \left(\frac{1}{4\alpha^4\beta^8} + \frac{64}{27\beta^6} \right)^{1/2} \right]^{1/3} \tag{33}$$

$$v = -\left[-\frac{1}{2\alpha^2\beta^4} + \left(\frac{1}{4\alpha^4\beta^8} + \frac{64}{27\beta^6} \right)^{1/2} \right]^{1/3} \tag{34}$$

and

$$y = \left[\left(\frac{1}{4\alpha^4\beta^8} + \frac{64}{27\beta^6} \right)^{1/2} + \frac{1}{2\alpha^2\beta^4} \right]^{1/3} - \left[\left(\frac{1}{4\alpha^4\beta^8} + \frac{64}{27\beta^6} \right)^{1/2} - \frac{1}{2\alpha^2\beta^4} \right]^{1/3} \tag{35}$$

where positive cube roots are chosen in both cases. It is clear that $y > 0$ so

$$e = (y)^{1/2} \tag{36}$$

$$f = -\left(\frac{y^2}{4} + \frac{1}{\beta^2} \right)^{1/2} \tag{37}$$

the sign of f being negative since

$$-c + \frac{ay}{2} = -\frac{1}{\alpha\beta^2} \tag{38}$$

Then

$$r^2 + \frac{y}{2} = \pm \left[(y)^{1/2} r - \left(\frac{y^2}{4} + \frac{1}{\beta^2} \right)^{1/2} \right] \tag{39}$$

or, with signs correspondingly placed above and below,

$$r^2 \mp (y)^{1/2} r + \frac{y}{4} = \mp \left(\frac{y^2}{4} + \frac{1}{\beta^2} \right)^{1/2} - \frac{y}{4} \tag{40}$$

Since the left member of equation (40) is non-negative, the top signs must be discarded. The result is (since r cannot be negative)

$$r = -\frac{(y)^{1/2}}{2} + \left[\left(\frac{y^2}{4} + \frac{1}{\beta^2} \right)^{1/2} - \frac{y}{4} \right]^{1/2} \tag{41}$$

Equations (35) and (41) together define r as a function of α and β.

Applying Uspensky's solution (Ref. 3) to equation (14),

$$a = 0, \qquad b = -2, \qquad c = -\frac{1}{\alpha^2\beta}, \qquad d = 1 \tag{42}$$

so

$$A = 2, \qquad B = -4, \qquad C = -8 - \frac{1}{\alpha^4\beta^2} \tag{43}$$

and

$$p = -\frac{16}{3}, \qquad q = -\frac{128}{27} - \frac{1}{\alpha^4\beta^2} \tag{44}$$

Then

$$u = \left\{ \left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2} \right) + \left[\left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2} \right)^2 - \left(\frac{16}{9} \right)^3 \right]^{1/2} \right\}^{1/3} \tag{45}$$

and

$$v = \left\{ \left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2} \right) - \left[\left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2} \right)^2 - \left(\frac{16}{9} \right)^3 \right]^{1/2} \right\}^{1/3} \tag{46}$$

where positive cube roots are chosen in both cases.

Then, since

$$\left(\frac{64}{27}\right)^2 = \left(\frac{16}{9}\right)^3$$

$$z = \frac{4}{3}\left(\left\{\left(1+\frac{27}{128\alpha^4\beta^2}\right)+\left[\left(1+\frac{27}{128\alpha^4\beta^2}\right)^2-1\right]^{1/2}\right\}^{1/3}+\left\{\left(1+\frac{27}{128\alpha^4\beta^2}\right)-\left[\left(1+\frac{27}{128\alpha^4\beta^2}\right)^2-1\right]^{1/2}\right\}^{1/3}\right) \quad (47)$$

Since

$$\left\{\left(1+\frac{27}{128\alpha^4\beta^2}\right)+\left[\left(1+\frac{27}{128\alpha^4\beta^2}\right)^2-1\right]^{1/2}\right\}^{1/3}\left\{\left(1+\frac{27}{128\alpha^4\beta^2}\right)-\left[\left(1+\frac{27}{128\alpha^4\beta^2}\right)^2-1\right]^{1/2}\right\}^{1/3} = 1 \quad (48)$$

it is clear that equation (47) can be written as

$$z = \frac{4}{3}\left(t + \frac{1}{t}\right) \quad (49)$$

where $t > 0$. Since the minimum value of $t + 1/t$ for positive t is 2, the minimum value of z is 8/3 and the corresponding minimum value of y, given according to equation (23) by

$$y = z - \frac{2}{3} \quad (50)$$

is 2. Then

$$e = (2 + y)^{1/2} \quad (51)$$

$$f = +\left(\frac{y^2}{4} - 1\right)^{1/2} \quad \text{if } \beta > 0 \quad (52)$$

$$= -\left(\frac{y^2}{4} - 1\right)^{1/2} \quad \text{if } \beta < 0$$

since

$$-c + \frac{ay}{2} = \frac{1}{\alpha^2\beta} \quad (53)$$

Then, if $\beta > 0$,

$$\sin^2\theta + \frac{y}{2} = \pm\left[(2+y)^{1/2}\sin\theta + \left(\frac{y^2}{4}-1\right)^{1/2}\right] \quad (54)$$

or, with signs correspondingly placed above and below,

$$\sin^2\theta \mp (2+y)^{1/2}\sin\theta + \frac{2+y}{4} = \pm\left(\frac{y^2}{4}-1\right)^{1/2} + \frac{1}{2} - \frac{y}{4} \quad (55)$$

Since the minimum value of y is 2, and since the left member of equation (55) is non-negative, the bottom signs must be discarded. The result is (for $\beta > 0$), since $\sin\theta$ cannot exceed 1,

$$\sin\theta = \frac{(2+y)^{1/2}}{2} - \left[\left(\frac{y^2}{4}-1\right)^{1/2} + \frac{1}{2} - \frac{y}{4}\right]^{1/2} \quad (56)$$

If $\beta < 0$,

$$\sin^2\theta + \frac{y}{2} = \pm\left[(2+y)^{1/2}\sin\theta - \left(\frac{y^2}{4}-1\right)^{1/2}\right] \quad (57)$$

or, with signs correspondingly placed above and below,

$$\sin^2\theta \mp (2+y)^{1/2}\sin\theta + \frac{2+y}{4} = \mp\left(\frac{y^2}{4}-1\right)^{1/2} + \frac{1}{2} - \frac{y}{4} \quad (58)$$

Since the minimum value of y is 2 and since the left member of equation (58) is non-negative, the top signs must be discarded. The result is (for $\beta < 0$), since $\sin\theta$ must not be less than -1,

$$\sin\theta = -\frac{(2+y)^{1/2}}{2} + \left[\left(\frac{y^2}{4}-1\right)^{1/2} + \frac{1}{2} - \frac{y}{4}\right]^{1/2} \quad (59)$$

Equations (47), (50), (56), and (59) together define $\sin\theta$ as a function of α and β.

Summarizing, the transformation from dipolar coordinates to spherical coordinates is effected as follows:

First of all, of course,

$$\phi = \phi$$

If $\alpha = 0$, β is undefined. Then $r = 0$ and θ is undefined.

If $\alpha > 0$, $\beta = 0$, then $\theta = 0$ and $r = \alpha$.

If $\alpha > 0$, $\beta \neq 0$,

$$r = -\frac{(y)^{1/2}}{2} + \left[\left(\frac{y^2}{4} + \frac{1}{\beta^2}\right)^{1/2} - \frac{y}{4}\right]^{1/2}$$

where

$$y = \left[\left(\frac{1}{4\alpha^4\beta^8} + \frac{64}{27\beta^6}\right)^{1/2} + \frac{1}{2\alpha^2\beta^4}\right]^{1/3} - \left[\left(\frac{1}{4\alpha^4\beta^8} + \frac{64}{27\beta^6}\right)^{1/2} - \frac{1}{2\alpha^2\beta^4}\right]^{1/3}$$

and

$$\theta = \arcsin x \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

where

$$x = \frac{(2+y)^{1/2}}{2} - \left[\left(\frac{y^2}{4} - 1\right)^{1/2} + \frac{1}{2} - \frac{y}{4}\right]^{1/2} \qquad \text{if } \beta > 0$$

$$= -\frac{(2+y)^{1/2}}{2} + \left[\left(\frac{y^2}{4} - 1\right)^{1/2} + \frac{1}{2} - \frac{y}{4}\right]^{1/2} \qquad \text{if } \beta < 0$$

and

$$y = \left\{\left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2}\right) + \left[\left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2}\right)^2 - \left(\frac{64}{27}\right)^2\right]^{1/2}\right\}^{1/3} + \left\{\left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2}\right) - \left[\left(\frac{64}{27} + \frac{1}{2\alpha^4\beta^2}\right)^2 - \left(\frac{64}{27}\right)^2\right]^{1/2}\right\}^{1/3} - \frac{2}{3}$$

IV. A NUMERICAL METHOD FOR TRANSFORMING TO SPHERICAL COORDINATES

In Section III, a straightforward analytical formulation has been given to effect the transformation from dipolar coordinates to spherical coordinates.

If the only goal were to determine r, θ values corresponding to a given set of α, β values, and if a high-speed computing facility were available, it might be desirable to proceed as follows:

A. Does $\alpha = 0$? If yes, then $r = 0$ and θ is indeterminate. If no, proceed to next step.

B. Does $\beta = 0$? If yes, then $\theta = 0$ and $r = \alpha$. If no, then r and θ can be determined from the following two steps.

C. To determine r, note that the maximum value which r might have is α. Choose $r_1 = \alpha$ and $r_2 = \alpha/2$. Evaluate

$$f(r) \equiv r^4 + \frac{1}{\alpha\beta^2} r - \frac{1}{\beta^2}$$

for r_1 and r_2. If $f(r_1)$ and $f(r_2)$ have the same sign, choose $r_3 = r_2/2$. Continue choosing $r_k = r_{k-1}/2$ until $f(r_k)$ and $f(r_{k-1})$ have opposite signs. As soon as this occurs, choose

$$r_{k+1} = \frac{r_k + r_{k-1}}{2}$$

For subsequent choices, use the value halfway between r_{k+1} and the corresponding r_k or r_{k-1}, which makes $f(r_k)f(r_{k+1})$ or $f(r_{k-1})f(r_{k+1})$ have a negative sign. The process is terminated at $r = r_n$ when $|f(r_n)|$ is less than some preassigned positive quantity.

D. To determine θ, note that $0 < \theta < \pi/2$ when $\beta > 0$ and $-\pi/2 < \theta < 0$ when $\beta < 0$. Choose $x_1 = 0$ and $x_2 = 1$. Evaluate

$$g(x) \equiv x^4 - 2x^2 - \frac{x}{\alpha^2|\beta|} + 1$$

for x_1 and x_2. Choose

$$x_3 = \frac{x_1 + x_2}{2}$$

For subsequent choices, use the value halfway between x_{k+1} and the corresponding x_k or x_{k-1}, which makes $g(x_k)g(x_{k+1})$ or $g(x_{k-1})g(x_{k+1})$ have a negative sign. The process is terminated at $x = x_n$ when $|g(x_n)|$ is less than some preassigned positive quantity. Then

$$\theta = \frac{\beta}{|\beta|} \arcsin x$$

where the principal-valued range $(-\pi/2, \pi/2)$ is used for the arc sin function.

V. BASIC VECTOR FORMULAS IN DIPOLAR COORDINATES

Dipolar coordinates are but a special case of generalized orthogonal curvilinear coordinates and are probably best described in terms of that framework.

The quantities of particular interest in most applications are to be found among those commonly described by

$$\nabla\Phi,\ \nabla^2\Phi,\ \nabla\cdot\mathbf{F},\ \nabla\times\mathbf{F},\ (\mathbf{A}\cdot\nabla)\,\Phi,\ (\mathbf{A}\cdot\nabla)\,\mathbf{F},\quad \text{and } \nabla^2\mathbf{F} \tag{60}$$

where Φ is a scalar function and $\mathbf{A}$ and $\mathbf{F}$ are vector functions. Hoffman (Ref. 2) expressed some of these in terms of a mixed set of coordinates (his θ is the complement of that used in this Report) but omitted $(\mathbf{A}\cdot\nabla)\,\Phi$, $(\mathbf{A}\cdot\nabla)\,\mathbf{F}$, and $\nabla^2\mathbf{F}$. He furthermore left unsolved the matter of expressing r and θ in terms of α and β.

Use of the symbol ∇ can lead to considerable confusion. In particular, the use of the same symbol for $\nabla^2\mathbf{F}$ as for $\nabla^2\Phi$ leads to sufficient confusion that it has been considered desirable to prepare a separate discussion of the matter, which is given in the Appendix.

In the following, it is assumed that the reader is familiar with the metric

$$ds^2 = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2 \tag{61}$$

as well as the formulas in the orthogonal right-handed coordinate system u_1, u_2, u_3 for grad Φ, div $\mathbf{F}$, and curl $\mathbf{F}$,

$$\operatorname{grad}\Phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3} \tag{62}$$

$$\operatorname{div}\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_2h_1F_3)\right] \tag{63}$$

$$\begin{aligned}\operatorname{curl}\mathbf{F} = {} & \frac{\mathbf{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right] \\ & + \frac{\mathbf{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_3F_3)\right] \\ & + \frac{\mathbf{e}}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right]\end{aligned} \tag{64}$$

where

$$\mathbf{F} = \mathbf{e}_1F_1 + \mathbf{e}_2F_2 + \mathbf{e}_3F_3 \tag{65}$$

and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors in the direction of increasing u_1, u_2, u_3, respectively, from the point P: (u_1, u_2, u_3).

The quantities (60) are more properly described by the identities

$$\nabla\Phi \equiv \operatorname{grad}\Phi \tag{66}$$

$$\nabla^2\Phi \equiv \operatorname{div}(\operatorname{grad}\Phi) \tag{67}$$

$$\nabla\cdot\mathbf{F} \equiv \operatorname{div}\mathbf{F} \tag{68}$$

$$\nabla\times\mathbf{F} \equiv \operatorname{curl}\mathbf{F} \tag{69}$$

$$(\mathbf{A}\cdot\nabla)\,\Phi \equiv \mathbf{A}\cdot(\operatorname{grad}\Phi) \tag{70}$$

$$\begin{aligned}(\mathbf{A}\cdot\nabla)\,\mathbf{F} \equiv \frac{1}{2}[& \operatorname{grad}(\mathbf{A}\cdot\mathbf{F}) - \operatorname{curl}(\mathbf{A}\times\mathbf{F}) \\ & - \mathbf{A}\times(\operatorname{curl}\mathbf{F}) - \mathbf{F}\times(\operatorname{curl}\mathbf{A}) \\ & + \mathbf{A}\operatorname{div}\mathbf{F} - \mathbf{F}\operatorname{div}\mathbf{A}]\end{aligned} \tag{71}$$

$$\nabla^2\mathbf{F} \equiv \operatorname{grad}(\operatorname{div}\mathbf{F}) - \operatorname{curl}(\operatorname{curl}\mathbf{F}) \tag{72}$$

In the special case that $\mathbf{A} = \mathbf{F}$ in identity (71), one has

$$(\mathbf{A}\cdot\nabla)\,\mathbf{A} \equiv \frac{1}{2}\operatorname{grad}(\mathbf{A}\cdot\mathbf{A}) - \mathbf{A}\times(\operatorname{curl}\mathbf{A}) \tag{71a}$$

Since, writing

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 \tag{73}$$

and

$$\mathbf{B} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3 \tag{74}$$

the quantities $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ are given, respectively, by

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 \tag{75}$$

and

$$\begin{aligned}(\mathbf{A}\times\mathbf{B}) = {} & (A_2B_3 - A_3B_2)\,\mathbf{e}_1 \\ & + (A_3B_1 - A_1B_3)\,\mathbf{e}_2 + (A_1B_2 - A_2B_1)\,\mathbf{e}_3\end{aligned} \tag{76}$$

repeated application of formulas (62), (63), (64), (75), and (76) will enable expressing any of the quantities (60) in terms of the h_1, h_2, h_3 (and their derivatives with respect to u_1, u_2, u_3) and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

There remains, then, but to identify α, ϕ, β, h_α, h_ϕ, h_β, and $\mathbf{e}_\alpha$, $\mathbf{e}_\phi$, $\mathbf{e}_\beta$ with u_1, u_2, u_3, h_1, h_2, h_3 and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively, and to determine h_α, h_ϕ, and h_β in terms of α, ϕ, and β. For the sake of brevity, h_α, h_ϕ, and h_β will actually be expressed in terms of r, $\sin\theta$, and $\cos\theta$. Since, for the range of values considered,

$$\cos\theta = (1 - \sin^2\theta)^{1/2} \tag{77}$$

and since Section III has shown how to express r and $\sin\theta$ in terms of α and β, it is clear that h_α, h_ϕ, and h_β will then have been defined analytically as functions of α and β.

VI. DETERMINATION OF h_α, h_ϕ, h_β

If the coordinates α, ϕ, β are known in terms of the rectangular coordinates x, y, z, the metric coefficients h_α, h_ϕ, h_β may be determined from

$$\begin{aligned}
\frac{1}{h_\alpha} &= \left[\left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2\right]^{1/2} \\
\frac{1}{h_\phi} &= \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]^{1/2} \\
\frac{1}{h_\beta} &= \left[\left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right]^{1/2}
\end{aligned} \tag{78}$$

Since α, ϕ, β are given as functions of r, ϕ, θ by equations (11), and since

$$\begin{aligned}
x &= r\cos\theta\cos\phi \\
y &= r\cos\theta\sin\phi \\
z &= r\sin\theta
\end{aligned} \tag{79}$$

and

$$\begin{aligned}
r &= (x^2 + y^2 + z^2)^{1/2} \\
\phi &= \arctan\frac{y}{x} \\
\theta &= \arcsin\frac{z}{(x^2 + y^2 + z^2)^{1/2}}
\end{aligned} \tag{80}$$

the quantities

$$\frac{\partial\alpha}{\partial x}, \frac{\partial\alpha}{\partial y}, \frac{\partial\alpha}{\partial z}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}, \frac{\partial\beta}{\partial x}, \frac{\partial\beta}{\partial y}, \frac{\partial\beta}{\partial z} \tag{81}$$

can be determined from relations of the type

$$\frac{\partial\alpha}{\partial x} = \frac{\partial\alpha}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\alpha}{\partial\phi}\frac{\partial\phi}{\partial x} + \frac{\partial\alpha}{\partial\theta}\frac{\partial\theta}{\partial x} \tag{82}$$

by expressing

$$\frac{\partial\alpha}{\partial r}, \frac{\partial\alpha}{\partial\phi}, \frac{\partial\alpha}{\partial\theta}, \frac{\partial r}{\partial x}, \frac{\partial\phi}{\partial x}, \frac{\partial\theta}{\partial x}, \text{etc.}$$

in terms of the coordinates r, ϕ, θ. Thus, for instance, from equations (11),

$$\begin{aligned}
\frac{\partial\alpha}{\partial r} &= \frac{1}{\cos^2\theta} \\
\frac{\partial\alpha}{\partial\phi} &= 0 \\
\frac{\partial\alpha}{\partial\theta} &= \frac{2r\sin\theta}{\cos^3\theta}
\end{aligned} \tag{83}$$

while, from equations (79) and (80),

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \cos\theta\cos\phi \\
\frac{\partial\phi}{\partial x} &= -\frac{\sin\phi}{r\cos\theta} \\
\frac{\partial\theta}{\partial x} &= -\frac{\sin\theta\cos\phi}{r}
\end{aligned} \tag{84}$$

which lead to

$$\frac{\partial\alpha}{\partial x} = \frac{1 - 3\sin^2\theta}{\cos^3\theta}\cos\phi \tag{85}$$

In a similar manner, one obtains

$$\frac{\partial\alpha}{\partial y} = \frac{1 - 3\sin^2\theta}{\cos^3\theta}\sin\phi$$
$$\frac{\partial\alpha}{\partial z} = \frac{3\sin\theta\cos\theta}{\cos^3\theta} \tag{86}$$

which then yield

$$h_\alpha = \frac{\cos^3\theta}{(1 + 3\sin^2\theta)^{1/2}} \tag{87}$$

Following a parallel procedure enables one to determine that

$$h_\phi = r\cos\theta \tag{88}$$

$$h_\beta = \frac{r^3}{(1 + 3\sin^2\theta)^{1/2}} \tag{89}$$

It must be emphasized again that, for the range $-\pi/2 \leqq \theta \leqq \pi/2$, $\cos\theta = (1 - \sin^2\theta)^{1/2}$, and that analytical expressions for r and $\sin\theta$ in terms of α and β have been given in Section III, so that h_α, h_ϕ, and h_β have now been determined in α, ϕ, and β coordinates.

VII. CONCLUDING REMARKS

As a result of this study, it can be seen that the introduction of dipolar coordinates leads to a very unwieldy analytical description for the commonly used vector quantities. It would therefore appear that another approach might be more fruitful in attempting to solve magnetohydrodynamic problems in which a dipole field is assumed. The difficulties of working with such problems involving dipole fields are not small, and one may perhaps conjecture the existence of another "law of conservation," which might be referred to as the "conservation of difficulty," and might possibly be expressed as follows:

"Difficulty in theoretical physical problems is conserved under all coordinate transformations."

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APPENDIX

On the Meaning of $\nabla^2\mathbf{F}$

ABSTRACT

In order to remove the confusion that exists regarding the meaning of $\nabla^2\mathbf{F}$, the notations $\nabla_1^2\Phi \equiv \text{div (grad } \Phi)$ and $\nabla_2^2\mathbf{F} \equiv \text{grad (div } \mathbf{F}) - \text{curl (curl } \mathbf{F})$ are introduced. The source of the confusion is discussed, and the relation between the quantities $\nabla_1^2\Phi$ and $\nabla_2^2\mathbf{F}$ is clarified without the introduction of tensor formalism. Illustrative examples are given using familiar coordinate systems.

I. INTRODUCTION

Considerable confusion exists in the literature concerned with vector fields as to the meaning of

$$\nabla^2\mathbf{F} \tag{A-1}$$

where $\mathbf{F}$ is a vector.

To cite but a few examples:

Coulson (Ref. A-1) states:

> Now curl curl $\equiv$ grad div $-\nabla^2$, so by combining (22)–(24) we obtain the standard differential equation for $\mathbf{A}$:
>
> $$\nabla^2\mathbf{A} = -4\pi\mathbf{J}$$
>
> This is a vector equation, each component of which resembles Poisson's equation, and may be solved in the same way.

No mention is made of any restriction to rectangular coordinates.

Hoffman (Ref. A-2) writes a vector equation

$$\nabla^2\mathbf{E} + k_0^2\,\boldsymbol{\kappa}\cdot\mathbf{E} = 0$$

($\boldsymbol{\kappa}$ represents a tensor of second rank) and then, using components corresponding to dipolar coordinates, E_α, E_ϕ, E_β, treats $\nabla^2\mathbf{E}$ as though its components were $\nabla^2 E_\alpha$, $\nabla^2 E_\phi$, $\nabla^2 E_\beta$.

Corson and Lorrain (Ref. A-3) state:

> It is correct to write
>
> $$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$
>
> in Cartesian coordinates but not in curvilinear coordinates.

Sommerfeld (Ref. A-4), writing on electrodynamics, refers to his equation (2)

$$\text{curl curl}\,\mathbf{E} = \text{grad div}\,\mathbf{E} - \nabla^2\mathbf{E}$$

with this statement:

> On the other hand, Eq. (2), according to the above, is restricted to the Cartesian coordinates x, y, z and the components E_x, E_y, E_z, since only these may be treated as scalar quantities.

The last example above is surprising because Sommerfeld (Ref. A-5), writing on mechanics of deformable bodies, defines $\nabla^2\mathbf{A}$ for general orthogonal coordinates by

$$\nabla^2\mathbf{A} = \text{grad div}\,\mathbf{A} - \text{curl curl}\,\mathbf{A}$$

The meaning of $\nabla^2\mathbf{F}$ is quite clear from the definition (Morse and Feshbach, Ref. A-6; Sommerfeld, Ref. A-5)

$$\nabla^2\mathbf{F} \equiv \text{grad}\,(\text{div}\,\mathbf{F}) - \text{curl}\,(\text{curl}\,\mathbf{F}) \tag{A-2}$$

which is sometimes written in the symbolic form

$$\nabla^2\mathbf{F} \equiv \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F}) \tag{A-2a}$$

Not infrequently, equation (A-2a) is written as

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F} \tag{A-2b}$$

as though defining $\nabla\times(\nabla\times\mathbf{F})$ in terms of an already known $\nabla^2\mathbf{F}$.

Such a procedure, as is shown in Section II of this Report, is largely the result of treating problems in terms of rectangular coordinates.

The expression

$$\nabla^2\Phi \tag{A-3}$$

where Φ is a scalar is defined by

$$\nabla^2\Phi \equiv \operatorname{div}(\operatorname{grad}\Phi) \tag{A-4}$$

or

$$\nabla^2\Phi \equiv \nabla\cdot(\nabla\Phi) \tag{A-4a}$$

The use of the same symbol ∇^2 is unfortunate, considering the different definitions (A-2) and (A-4). For the sake of further discussion, expressions (A-1) and (A-3) will be replaced by

$$\nabla_2^2\mathbf{F} \tag{A-5}$$

and

$$\nabla_1^2\Phi \tag{A-6}$$

where

$$\nabla_1^2\Phi \equiv \operatorname{div}(\operatorname{grad}\Phi) \tag{A-7}$$

and

$$\nabla_2^2\mathbf{F} \equiv \operatorname{grad}(\operatorname{div}\mathbf{F}) - \operatorname{curl}(\operatorname{curl}\mathbf{F}) \tag{A-8}$$

A vector-operator significance is frequently associated with the symbol ∇, again largely as a result of treating problems in terms of rectangular coordinates. It should be emphasized that a unique vector-operator significance for the symbol ∇ does not exist in generalized orthogonal coordinate systems.

The need to treat vector-field problems in coordinates other than rectangular appears to be sufficient reason for a clarification of the entire matter. Since, historically, the concepts were first developed with reference to rectangular coordinates, the subject is first reviewed from that point of view. Subsequently, the approach is made using generalized orthogonal coordinates, followed by examples using some familiar coordinate systems.

II. RECTANGULAR COORDINATES

In rectangular coordinates, the gradient of a scalar, and the divergence and the curl of a vector are defined by

$$\operatorname{grad}\psi = \mathbf{i}\frac{\partial\psi}{\partial x} + \mathbf{j}\frac{\partial\psi}{\partial y} + \mathbf{k}\frac{\partial\psi}{\partial z} \tag{A-9}$$

$$\operatorname{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{A-10}$$

$$\operatorname{curl}\mathbf{A} = i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{A-11}$$

where

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z \tag{A-12}$$

Here, **i**, **j**, **k** are unit vectors in the x, y, z directions, respectively, and ψ, A_x, A_y, A_z are scalar functions of x, y, z.

Thus

$$\nabla_1^2\Phi = \operatorname{div}\left(\mathbf{i}\frac{\partial\Phi}{\partial x} + \mathbf{j}\frac{\partial\Phi}{\partial y} + \mathbf{k}\frac{\partial\Phi}{\partial z}\right) \tag{A-13}$$

or

$$\nabla_1^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \tag{A-14}$$

On the other hand,

$$\begin{aligned}\operatorname{grad}(\operatorname{div}\mathbf{F}) &= \operatorname{grad}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\\ &= \mathbf{i}\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x\partial y} + \frac{\partial^2 F_z}{\partial x\partial z}\right)\\ &\quad + \mathbf{j}\left(\frac{\partial^2 F_x}{\partial y\partial x} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y\partial z}\right)\\ &\quad + \mathbf{k}\left(\frac{\partial^2 F_x}{\partial z\partial x} + \frac{\partial^2 F_y}{\partial z\partial y} + \frac{\partial^2 F_z}{\partial z^2}\right)\end{aligned} \tag{A-15}$$

while

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}\mathbf{F}) &= \operatorname{curl}\left[\mathbf{i}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)+\mathbf{j}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)+\mathbf{k}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\right]\\
&= \mathbf{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\right]\\
&\quad+\mathbf{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\right]\\
&\quad+\mathbf{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\right]\end{aligned}\tag{A-16}$$

Thus

$$\begin{aligned}\nabla_2^2\mathbf{F} &= \mathbf{i}\left(\frac{\partial^2 F_x}{\partial x^2}+\frac{\partial^2 F_x}{\partial y^2}+\frac{\partial^2 F_x}{\partial z^2}\right)\\
&\quad+\mathbf{j}\left(\frac{\partial^2 F_y}{\partial x^2}+\frac{\partial^2 F_y}{\partial y^2}+\frac{\partial^2 F_y}{\partial z^2}\right)\\
&\quad+\mathbf{k}\left(\frac{\partial^2 F_z}{\partial x^2}+\frac{\partial^2 F_z}{\partial y^2}+\frac{\partial^2 F_z}{\partial z^2}\right)\end{aligned}\tag{A-17}$$

or

$$\nabla_2^2\mathbf{F} = \mathbf{i}\ \nabla_1^2 F_x + \mathbf{j}\ \nabla_1^2 F_y + \mathbf{k}\ \nabla_1^2 F_z \tag{A-18}$$

It is tempting at this point to look at ∇_1^2 and ∇_2^2 as operational symbols having separate entities, with both appearing in the present case to be

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{A-19}$$

Herein arises the confusion. If one yields to this temptation and uses ∇^2 to represent both, then equation (A-18), which can be written correctly as

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{F} &= \mathbf{i}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_x\\
&\quad+\mathbf{j}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_y\\
&\quad+\mathbf{k}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_z\end{aligned}\tag{A-20}$$

appears to have the form

$$\nabla^2(\mathbf{F}) = \mathbf{i}\ \nabla^2(F_x) + \mathbf{j}\ \nabla^2(F_y) + \mathbf{k}\ \nabla^2(F_z) \tag{A-21}$$

or suppressing parentheses and adding to the confusion,

$$\nabla^2\mathbf{F} = \mathbf{i}\ \nabla^2 F_x + \mathbf{j}\ \nabla^2 F_y + \mathbf{k}\ \nabla^2 F_z \tag{A-22}$$

If $\nabla^2\mathbf{F}$ is written in the form of equation (A-22), it must be understood that ∇^2 simply stands for the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. It just happens that in rectangular coordinates

$$\nabla_1^2\Phi = \nabla^2(\Phi)$$

and

$$\nabla_2^2\mathbf{F} = \mathbf{i}\ \nabla^2(F_x) + \mathbf{j}\ \nabla^2(F_y) + \mathbf{k}\ \nabla^2(F_z) \tag{A-23}$$

If one restricts oneself to rectangular coordinates, equation (A-23) might be taken as a definition of $\nabla_2^2\mathbf{F}$. In this case, curl (curl **F**) could then be defined by

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla_2^2\mathbf{F} \tag{A-24}$$

Such an approach for those who face vector-field problems in terms of other coordinate systems leads to incorrect results.

III. ORTHOGONAL CURVILINEAR COORDINATES

The difference between the concepts $\nabla_1^2\Phi$ and $\nabla_2^2\mathbf{F}$ is made very clear if one resorts to tensor formalism (see Morse and Feshbach, Ref. A-6). In the present treatment, however, such formalism will be avoided as an unnecessary complication.

The metric in the orthogonal curvilinear coordinates u_1, u_2, u_3 is given by

$$ds^2 = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2 \tag{A-25}$$

where, in rectangular coordinates,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{A-26}$$

It is assumed that transformations are known in the form

$$\begin{aligned} x &= x\,(u_1, u_2, u_3) \\ y &= y\,(u_1, u_2, u_3) \\ z &= z\,(u_1, u_2, u_3) \end{aligned} \tag{A-27}$$

or

$$\begin{aligned} u_1 &= u_1\,(x, y, z) \\ u_2 &= u_2\,(x, y, z) \\ u_3 &= u_3\,(x, y, z) \end{aligned} \tag{A-28}$$

The h_i $(i = 1, 2, 3)$ can be determined from either

$$h_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \tag{A-29}$$

or

$$h_i^2 = \left[\left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial u_i}{\partial y}\right)^2 + \left(\frac{\partial u_i}{\partial z}\right)^2\right]^{-1} \tag{A-30}$$

Using $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ to represent unit vectors in the directions of increasing u_1, u_2, u_3, respectively, from the point P: (u_1, u_2, u_3), with the further assumption that the coordinates are labeled to make $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ a right-handed triad, the quantities grad ϕ, div $\mathbf{F}$, and curl $\mathbf{F}$ can be shown (Morse and Feshbach, Ref. A-6, or Lass, Ref. A-7) to have the following form:

$$\operatorname{grad}\Phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3} \tag{A-31}$$

$$\operatorname{div}\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_2h_1F_3)\right] \tag{A-32}$$

$$\begin{aligned} \operatorname{curl}\mathbf{F} = {} & \frac{\mathbf{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right] + \frac{\mathbf{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_3F_3)\right] \\ & + \frac{\mathbf{e}_3}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right] \end{aligned} \tag{A-33}$$

where

$$\mathbf{F} = \mathbf{e}_1F_1 + \mathbf{e}_2F_2 + \mathbf{e}_3F_3 \tag{A-34}$$

Using expressions (A-31) and (A-32) in definition (A-7) the result is

$$\nabla_1^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right] \tag{A-35}$$

Using expressions (A-31), (A-32), and (A-33) in definition (A-8), the first term on the right side gives

$$\begin{aligned}\operatorname{grad}(\operatorname{div}\mathbf{F}) = {} & \frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right]\right\} \\ & + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right]\right\} \\ & + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right]\right\}\end{aligned} \tag{A-36}$$

while the second term on the right side gives

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}\mathbf{F}) = {} & \frac{\mathbf{e}_1}{h_2h_3}\left\{\frac{\partial}{\partial u_2}\left[\frac{h_3}{h_1h_2}\left(\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right)\right] - \frac{\partial}{\partial u_3}\left[\frac{h_2}{h_3h_1}\left(\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_3F_3)\right)\right]\right\} \\ & + \frac{\mathbf{e}_2}{h_3h_1}\left\{\frac{\partial}{\partial u_3}\left[\frac{h_1}{h_2h_3}\left(\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right)\right] - \frac{\partial}{\partial u_1}\left[\frac{h_3}{h_1h_2}\left(\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right)\right]\right\} \\ & + \frac{\mathbf{e}_3}{h_1h_2}\left\{\frac{\partial}{\partial u_1}\left[\frac{h_2}{h_3h_1}\left(\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_3F_3)\right)\right] - \frac{\partial}{\partial u_2}\left[\frac{h_1}{h_2h_3}\left(\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right)\right]\right\}\end{aligned} \tag{A-37}$$

By subtracting the right side of equation (A-37) from the right side of equation (A-36), the result is the expression for $\nabla_2^2\mathbf{F}$ in orthogonal curvilinear coordinates:

$$\nabla_2^2\mathbf{F} = (\nabla_2^2\mathbf{F})_1\mathbf{e}_1 + (\nabla_2^2\mathbf{F})_2\mathbf{e}_2 + (\nabla_2^2\mathbf{F})_3\mathbf{e}_3 \tag{A-38}$$

where

$$\begin{aligned}(\nabla_2^2\mathbf{F})_1 = {} & \frac{1}{h_1}\frac{\partial}{\partial u_1}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right]\right\} \\ & - \frac{1}{h_2h_3}\left\{\frac{\partial}{\partial u_2}\left[\frac{h_3}{h_1h_2}\left(\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right)\right] - \frac{\partial}{\partial u_3}\left[\frac{h_2}{h_3h_1}\left(\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_3F_3)\right)\right]\right\}\end{aligned} \tag{A-39}$$

$$\begin{aligned}(\nabla_2^2\mathbf{F})_2 = {} & \frac{1}{h_2}\frac{\partial}{\partial u_2}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right]\right\} \\ & - \frac{1}{h_3h_1}\left\{\frac{\partial}{\partial u_3}\left[\frac{h_1}{h_2h_3}\left(\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right)\right] - \frac{\partial}{\partial u_1}\left[\frac{h_3}{h_1h_2}\left(\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right)\right]\right\}\end{aligned} \tag{A-40}$$

$$\begin{aligned}(\nabla_2^2\mathbf{F})_3 = {} & \frac{1}{h_3}\frac{\partial}{\partial u_3}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right]\right\} \\ & - \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial u_1}\left[\frac{h_2}{h_3h_1}\left(\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_3F_3)\right)\right] - \frac{\partial}{\partial u_2}\left[\frac{h_1}{h_2h_3}\left(\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right)\right]\right\}\end{aligned} \tag{A-41}$$

From a comparison of equations (A-35), (A-38), (A-39), (A-40), (A-41), the difference between $\nabla_1^2\phi$ and $\nabla_2^2\mathbf{F}$ is quite clear. Note that, in general, each component of $\nabla_2^2\mathbf{F}$ depends on all three components of $\mathbf{F}$.

IV. ILLUSTRATIVE EXAMPLES

The difficulty of working with $\nabla_2^2\mathbf{F}$ varies with the choice of coordinate system. Unfortunately, the choice of coordinate system is at times strongly influenced by boundary conditions or the geometry of a particular problem under consideration.

Three examples follow, using familiar geometries:

Rectangular Coordinates

In this case

$$u_1 = x, \quad u_2 = y, \quad u_3 = z$$

$$h_1 = h_2 = h_3 = 1$$

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}$$

$$F_1 = F_x, \quad F_2 = F_y, \quad F_3 = F_z$$

Then equation (A-35) becomes

$$\nabla_1^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \qquad \text{(A-42)}$$

and equation (A-38) becomes

$$\nabla\mathbf{F}_2^2 = (\nabla_1^2 F_x)\,\mathbf{i} + (\nabla_1^2 F_y)\,\mathbf{j} + (\nabla_1^2 F_z)\,\mathbf{k} \qquad \text{(A-43)}$$

Cylindrical Coordinates

In this case

$$u_1 = \rho = (x^2 + y^2)^{1/2},$$

$$u_2 = \phi = \begin{cases} \text{arc cos}\,[x/(x^2+y^2)^{1/2}] \\ \text{arc sin}\,[y/(x^2+y^2)^{1/2}] \end{cases}, \quad u_3 = z$$

$$h_1 = h_3 = 1, \quad h_2 = \rho$$

$$\mathbf{e}_1 = \mathbf{e}_\rho, \quad \mathbf{e}_2 = \mathbf{e}_\phi, \quad \mathbf{e}_3 = \mathbf{e}_z$$

$$F_1 = F_\rho, \quad F_2 = F_\phi, \quad F_3 = F_z$$

Then equation (A-35) becomes

$$\nabla_1^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} \qquad \text{(A-44)}$$

and equation (A-38) becomes

$$\begin{aligned}\nabla_2^2\mathbf{F} = &\left(\nabla_1^2 F_\rho - \frac{1}{\rho^2}F_\rho - \frac{2}{\rho^2}\frac{\partial F_\phi}{\partial\phi}\right)\mathbf{e}_\rho \\ &+ \left(\nabla_1^2 F_\phi - \frac{1}{\rho^2}F_\phi + \frac{2}{\rho^2}\frac{\partial F_\rho}{\partial\phi}\right)\mathbf{e}_\phi \\ &+ (\nabla_1^2 F_2)\mathbf{e}_2 \end{aligned} \qquad \text{(A-45)}$$

Spherical Coordinates

In this case

$$u_1 = r = (x^2 + y^2 + z^2)^{1/2},$$

$$u_2 = \theta = \text{arc cos}\,[z/(x^2 + y^2 + z^2)^{1/2}]$$

$$u_3 = \phi = \begin{cases} \text{arc cos}\,[x/(x^2+y^2)^{1/2}] \\ \text{arc sin}\,[y/(x^2+y^2)^{1/2}] \end{cases}$$

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta$$

$$\mathbf{e}_1 = \mathbf{e}_r, \quad \mathbf{e}_2 = \mathbf{e}_\theta, \quad \mathbf{e}_3 = \mathbf{e}_\phi$$

$$F_1 = F_r, \quad F_2 = F_\theta, \quad F_3 = F_\phi$$

Then equation (A-35) becomes

$$\nabla_1^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \qquad \text{(A-46)}$$

and equation (A-38) becomes

$$\begin{aligned}\nabla_2^2\mathbf{F} = &\left[\nabla_1^2 F_r - \frac{2}{r^2}F_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot F_\theta) - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\right]\mathbf{e}_r \\ &+ \left[\nabla_1^2 F_\theta - \frac{1}{r^2\sin^2\theta}F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\right]\mathbf{e}_\theta \\ &+ \left[\nabla_1^2 F_\phi - \frac{1}{r^2\sin^2\theta}F_\phi + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\phi}\right]\mathbf{e}_\phi \end{aligned}$$

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